

THE RANK OF THE ENDOMORPHISM MONOID OF A PARTITION

JOÃO ARAÚJO AND CSABA SCHNEIDER

ABSTRACT. The rank of a semigroup is the cardinality of a smallest generating set. In this paper we compute the rank of the endomorphism monoid of a non-trivial uniform partition of a finite set, that is, the semigroup of those transformations of a finite set that leave a non-trivial uniform partition invariant. That involves proving that the rank of a wreath product of two symmetric groups is two and then use the fact that the endomorphism monoid of a partition is isomorphic to a wreath product of two full transformation semigroups. The calculation of the rank of these semigroups solves an open question.

1. INTRODUCTION

If S is a semigroup and U is a subset of S then we say that U *generates* S if every element of S is expressible as a word in the elements of U . We use the convention that the empty word represents the identity element. The *rank* of a semigroup S , denoted by $\text{rank } S$, is the minimum among the cardinalities of its generating sets. It is well-known that a finite full transformation semigroup has rank 3, while a finite full partial transformation semigroup has rank 4 (see [14, Exercises 1.9.7 and 1.9.13]). Similar results were proved for many different classes of transformation semigroups (such as total, partial, partial one-to-one, order preserving) and their ideals; see [4, 10, 15, 20, 22].

Some generalizations of the notion of rank, for instance the *idempotent rank* and the *nilpotent rank*, also attracted a great deal of attention (see [5, 9, 19], among others).

Finally, in recent years, the notion of relative rank has been subjected to extensive research (see for example [1, 7, 13, 11, 12, 16]). Relative rank is a useful notion when dealing with finite semigroups (see Lemma 3.1), and it is crucial when dealing with uncountable semigroups. In fact, in such semigroups, the notion of rank is not very informative as the rank and the cardinality coincide. To a large extent, this line of research was prompted by two old papers by Sierpiński [24] and by Banach [3] (see also [2]).

In this paper we deal with the endomorphism monoid of a non-trivial uniform partition. We prove that the rank of such semigroup is 4, thus settling a problem posed in [17]. We also calculate the ranks of some related transformation semigroups.

If X is a finite set then the set of transformations on X form the *full transformation monoid on X* and is denoted by $T(X)$. We assume that transformations act on the right; that is, if $x \in X$ and $f \in T(X)$, then xf denotes the image of x under

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f . Let \mathcal{P} be a partition of X ; that is, $\mathcal{P} = \{P_1, \dots, P_m\}$ where $P_1, \dots, P_m \subseteq X$, $P_i \cap P_j = \emptyset$ whenever $i \neq j$, and $X = P_1 \cup \dots \cup P_m$. The equivalence relation that corresponds to a partition \mathcal{P} is denoted $\sim_{\mathcal{P}}$. The elements of $T(X)$ that preserve the partition \mathcal{P} form a semigroup and is denoted by $T(X, \mathcal{P})$. Using symbols,

$$\begin{aligned} T(X, \mathcal{P}) &= \{f \in T(X) \mid (\forall P_i \in \mathcal{P})(\exists P_j \in \mathcal{P}) P_i f \subseteq P_j\} \\ &= \{f \in T(X) \mid \text{if } x \sim_{\mathcal{P}} y \text{ then } xf \sim_{\mathcal{P}} yf\}. \end{aligned}$$

The partition \mathcal{P} is called *uniform* if $|P_i| = |P_j|$ for all $i, j \in \{1, \dots, m\}$. The main result of Huisheng's paper [17] is that for a uniform partition \mathcal{P} the rank of $T(X, \mathcal{P})$ is at most 6, or smaller in some degenerate cases. Huisheng's proof relied on the observation that there is a strong relationship between rank $T(X, \mathcal{P})$ and the rank of the group G of invertible transformations in $T(X, \mathcal{P})$. He proved that the rank of G is at most 4 (or smaller in some degenerate cases). In the present paper, we are able to show, for $|X| \geq 3$, that $\text{rank } G = 2$ (see Theorem 4.1). In order to facilitate the proof of our results, we use the concept of wreath products of transformation semigroups; see Section 2.

We can also consider the rank of some further interesting semigroups related to a partition \mathcal{P} of a set X . Let us define

$$\begin{aligned} \Sigma(X, \mathcal{P}) &= \{f \in T(X) \mid x \sim_{\mathcal{P}} y \text{ if and only if } xf \sim_{\mathcal{P}} yf\}, \text{ and} \\ \Gamma(X, \mathcal{P}) &= \{f \in T(X, \mathcal{P}) \mid (\forall P_i, P_j \in \mathcal{P}) \text{ either } P_i f \cap P_j \neq \emptyset \text{ or } P_i f = P_j\}. \end{aligned}$$

It is routine to check that $\Sigma(X, \mathcal{P}), \Gamma(X, \mathcal{P}) \leq T(X, \mathcal{P})$.

The following theorem improves Huisheng's result and settles the problem of determining the rank of $T(X, \mathcal{P})$. It also gives the rank of $\Sigma(X, \mathcal{P})$ and $\Gamma(X, \mathcal{P})$. A partition of X is said to be trivial if it has 1 or $|X|$ parts (that is, the trivial partitions are the identity and the universal partitions).

Theorem 1.1. *If X is a finite set such that $|X| \geq 3$, and \mathcal{P} is a non-trivial uniform partition of X , then $\text{rank } T(X, \mathcal{P}) = 4$ and $\text{rank } \Sigma(X, \mathcal{P}) = \text{rank } \Gamma(X, \mathcal{P}) = 3$.*

2. WREATH PRODUCTS OF TRANSFORMATION SEMIGROUPS

We define the wreath product of two transformation semigroups following [25] (see also [21, Chapter 10]). The material of this section is well-known, however, we felt that it was necessary to present it in order to make the paper self-contained and also to set our system of notation. Let S and R be two transformation semigroups acting on the sets Y and Z , respectively, and let B denote the set of functions $f : Z \rightarrow S$. The underlying set of the wreath product $S \wr R$ is the Cartesian product $B \times R$ and to each element $(f, r) \in B \times R$, we assign a transformation of the set $Y \times Z$ defined by

$$(1) \quad (y, z)(f, r) = (yf(z), zr) \quad \text{for all } y \in Y, z \in Z, f \in B, r \in R.$$

It is easy to see that this assignment is injective and that the image of this assignment is closed under composition; that is, the image is a subsemigroup of $T(Y \times Z)$.

From now on we assume that the sets Y and Z are finite and that S and R are monoids, that is, they contain the identity transformation id . Let us assume without loss of generality that $Y = \{1, \dots, n\}$ and $Z = \{1, \dots, m\}$. In this case, a function $f : Z \rightarrow S$ can be represented as the m -tuple (s_1, \dots, s_m) where $f(i) = s_i$ for all $i \in \{1, \dots, m\}$. This defines a bijection between $B \times R$ and $S^m \times R$, and, in turn, between $S \wr R$ and $S^m \times R$. Therefore we may view an element of $S \wr R$ as a pair $((s_1, \dots, s_m), r)$ where $s_1, \dots, s_m \in S$ and $r \in R$. The element $((s_1, \dots, s_m), r)$

will more briefly be denoted by $(s_1, \dots, s_m)r$. Setting $s = (s_1, \dots, s_m)$ the same element can be written as sr . By (1), the action of $(s_1, \dots, s_m)r$ on a pair $(y, z) \in Y \times Z$ is given by

$$(2) \quad (y, z)(s_1, \dots, s_m)r = (ys_z, zr).$$

Let $S \leq T(Y)$ and $R \leq T(Z)$ be transformation monoids as above. Then S and R can naturally be embedded into the wreath product $S \wr R$. Indeed, we may consider the following submonoids of $S \wr R$:

$$(3) \quad \overline{S}_i = \{(\text{id}, \dots, \text{id}, \overset{i\text{-th component}}{s}, \text{id}, \dots, \text{id}) \mid \text{id} \in S \wr R \mid s \in S\},$$

and

$$(4) \quad \overline{R} = \{(\text{id}, \dots, \text{id})r \in S \wr R \mid r \in R\}.$$

Set, for $i = \{1, \dots, m\}$, $\overline{Y}_i = \{(y, i) \mid y \in Y\}$. Easy computation shows that the elements of \overline{S}_i leave the set \overline{Y}_i invariant, and \overline{S}_i , considered as a transformation monoid on \overline{Y}_i , is isomorphic to the transformation monoid S . Let us further define $\overline{Z} = \{\overline{Y}_1, \dots, \overline{Y}_m\}$. Then \overline{R} leaves \overline{Z} invariant, and \overline{R} , considered as a transformation monoid on \overline{Z} , is isomorphic to R . Since $\overline{S}_1 \times \dots \times \overline{S}_m \cong S^m$ and $\overline{R} \cong R$, we may consider S^m and R as submonoids of $S \wr R$.

Since R is assumed to be a transformation monoid on the set $Z = \{1, \dots, m\}$, we may define a homomorphism $\vartheta : R \rightarrow S^m$:

$$(r\vartheta)(s_1, \dots, s_m) = (s_{1r}, \dots, s_{mr}).$$

We note that the action of $\text{End}(S^m)$ on S^m is a left-action. The homomorphism ϑ is useful for expressing the operation in $S \wr R$. Indeed, let $s_1, s_2 \in S^m$ and $r_1, r_2 \in R$. Then, viewing s_1r_1 and s_2r_2 as elements of $S \wr R$, easy computation shows that

$$(5) \quad (s_1r_1)(s_2r_2) = (s_1(r_1\vartheta)s_2)(r_1r_2).$$

Therefore the wreath product $S \wr R$ can also be viewed as the semidirect product $S^m \rtimes R$ of S^m and R with respect to the homomorphism $\vartheta : R \rightarrow \text{End}(S^m)$ (see [21, page 186]). In particular, if r is an invertible element of R and $s \in S^m$, then, considering s and r as elements of $S \wr R$, equation (5) implies that

$$(6) \quad rsr^{-1} = (r\vartheta)s.$$

Therefore conjugation by an invertible element of R leaves S^m invariant, and the conjugation action of R is given by the homomorphism ϑ . Indeed, from the last displayed line we obtain that $s^r = r^{-1}sr = (r^{-1}\vartheta)s$. Note that the conjugation action of the element r is actually given by the endomorphism $r^{-1}\vartheta$. The reason for this is that we assumed that endomorphisms act on the left, while the usual definition makes the conjugation action a right-action. Further, if r is an invertible element of R and $i, j \in \{1, \dots, m\}$ such that $ir = j$ then (6) implies that $\overline{S}_i^r = (r^{-1}\vartheta)\overline{S}_i = \overline{S}_j$.

For a finite set X , the invertible elements of $T(X)$ form the symmetric group $S(X)$. If \mathcal{P} is a partition of X then set $S(X, \mathcal{P}) = T(X, \mathcal{P}) \cap S(X)$. The next lemma connects the semigroups related to a partition to wreath products.

Lemma 2.1. *Let $Y = \{1, \dots, n\}$ and $Z = \{1, \dots, m\}$, set $X = Y \times Z$, and let \mathcal{P} denote the partition $\{\{(1, 1), \dots, (n, 1)\}, \dots, \{(1, m), \dots, (n, m)\}\}$. Then*

$$(i) \quad T(X, \mathcal{P}) = T(Y) \wr T(Z);$$

- (ii) $\Sigma(X, \mathcal{P}) = T(Y) \text{ wr } S(Z)$;
- (iii) $\Gamma(X, \mathcal{P}) = S(Y) \text{ wr } T(Z)$;
- (iv) $S(X, \mathcal{P}) = S(Y) \text{ wr } S(Z)$.

Proof. Since the proofs of these statements are very similar to each other, we only verify assertion (ii). Let $(y_1, z_1), (y_2, z_2) \in Y \times Z$ and assume that $t_1, \dots, t_m \in T(Y)$ $s \in S(Z)$ so that $(t_1, \dots, t_m)s \in T(Y) \text{ wr } S(Z)$. Set $w = (t_1, \dots, t_m)s$. We have, for $i = 1, 2$, that

$$(y_i, z_i)w = (y_i, z_i)(t_1, \dots, t_m)s = (y_i t_{z_i}, z_i s).$$

Thus, if $(y_1, z_1) \sim_{\mathcal{P}} (y_2, z_2)$, then $z_1 = z_2$, and then $z_1 s = z_2 s$; hence, in this case, $(y_1, z_1)w \sim_{\mathcal{P}} (y_2, z_2)w$. Conversely, if $(y_1, z_1)w \sim_{\mathcal{P}} (y_2, z_2)w$, then $z_1 s = z_2 s$, which, using that s is invertible, gives that $z_1 = z_2$; therefore $(y_1, z_1) \sim_{\mathcal{P}} (y_2, z_2)$. This shows that $w \in \Sigma(X, \mathcal{P})$, and so $T(Y) \text{ wr } S(Z) \leq \Sigma(X, \mathcal{P})$.

Suppose now that $x \in \Sigma(X, \mathcal{P})$. Then the defining property of $\Sigma(X, \mathcal{P})$ implies that x induces a permutation on the set \mathcal{P} . Since there is a one-to-one correspondence between \mathcal{P} and $Z = \{1, \dots, m\}$, we obtain that x induces a permutation on Z . Let this permutation be s . For $i = 1, \dots, m$, let us define a transformation $t_i \in T(Y)$. Let $j \in \{1, \dots, m\}$ such that $ix = j$. Then, for all $k \in \{1, \dots, n\}$ there is some $l_k \in \{1, \dots, n\}$ such that $(k, i)x = (l_k, j)$. Let t_i be the transformation that maps k to l_k for all $k \in \{1, \dots, n\}$. Then routine computation shows that $x = (t_1, \dots, t_m)s$, and so $x \in T(Y) \text{ wr } S(Z)$. Thus, $\Sigma(X, \mathcal{P}) \leq T(Y) \text{ wr } S(Z)$, and so $T(Y) \text{ wr } S(Z) = \Sigma(X, \mathcal{P})$. \square

3. RELATIVE RANK OF SEMIGROUPS

Let $U \subseteq S$ be a subset of a semigroup S . The *relative rank* of S modulo U , denoted by $\text{rank}(S : U)$, is the minimum among the cardinalities of subsets V of S such that $S = \langle V \cup U \rangle$. The relative rank was introduced in [13]. The next lemma shows that the rank of a transformation semigroup is related to its relative rank modulo the unit group.

Lemma 3.1. *Let S be a finite transformation semigroup and let G be the group of units in S . If $U \subseteq S$ such that $\langle U \rangle = S$, then $\langle U \cap G \rangle = G$. In particular, $\text{rank } S = \text{rank}(S : G) + \text{rank } G$.*

Proof. It suffices to show that $G \leq \langle U \cap G \rangle$, and so suppose that $g \in G$. Since U is a generating set of S , we obtain that $g = u_1 u_2 \cdots u_r$ with some $u_1, \dots, u_r \in U$. Since g is invertible, we obtain that u_1, \dots, u_r must also be invertible, and so $u_1, \dots, u_r \in G$. Thus $g \in \langle U \cap G \rangle$, and hence the assertion follows. \square

Next we determine the relative ranks of $T(X, \mathcal{P})$, $\Sigma(X, \mathcal{P})$ and $\Gamma(X, \mathcal{P})$ modulo their unit groups.

Lemma 3.2. *If X is a finite set and \mathcal{P} is a uniform partition of X , then*

$$\text{rank}(T(X, \mathcal{P}) : S(X, \mathcal{P})) = 2,$$

and

$$\text{rank}(\Gamma(X, \mathcal{P}) : S(X, \mathcal{P})) = \text{rank}(\Sigma(X, \mathcal{P}) : S(X, \mathcal{P})) = 1.$$

Proof. We may suppose without loss of generality that $X = Y \times Z$ where $Y = \{1, \dots, n\}$, $Z = \{1, \dots, m\}$ and \mathcal{P} is the partition

$$\{(1, 1), \dots, (n, 1)\}, \dots, \{(1, m), \dots, (n, m)\}.$$

By Lemma 2.1, $T(X, \mathcal{P}) = T(Y) \text{ wr } T(Z)$, $\Sigma(X, \mathcal{P}) = T(Y) \text{ wr } S(Z)$, $\Gamma(X, \mathcal{P}) = S(Y) \text{ wr } T(Z)$, and $S(X, \mathcal{P}) = S(Y) \text{ wr } S(Z)$, and so it suffices to show that

$$(7) \quad \text{rank}(T(Y) \text{ wr } T(Z) : S(Y) \text{ wr } S(Z)) = 2$$

and

$$(8) \quad \text{rank}(T(Y) \text{ wr } S(Z) : S(Y) \text{ wr } S(Z)) = \text{rank}(S(Y) \text{ wr } T(Z) : S(Y) \text{ wr } S(Z)) = 1.$$

Let $\bar{\alpha}$ denote the transformation in $T(Y)$ such that $1\bar{\alpha} = 2$ and $i\bar{\alpha} = i$ for all $i \in \{2, \dots, n\}$. Then $T(Y) = \langle S(Y) \cup \{\bar{\alpha}\} \rangle$ (see [14, Exercise 1.9.7]). Set $\alpha = (\bar{\alpha}, \text{id}, \dots, \text{id})\text{id}$ and $\beta = (\text{id}, \dots, \text{id})\bar{\alpha}$. We claim that

$$(9) \quad T(Y) \text{ wr } S(Z) = \langle S(Y) \text{ wr } S(Z) \cup \{\alpha\} \rangle,$$

$$(10) \quad S(Y) \text{ wr } T(Z) = \langle S(Y) \text{ wr } S(Z) \cup \{\beta\} \rangle,$$

$$(11) \quad T(Y) \text{ wr } T(Z) = \langle S(Y) \text{ wr } S(Z) \cup \{\alpha, \beta\} \rangle.$$

For $i \in \{1, \dots, m\}$, let us define the following submonoids of $T(Y) \text{ wr } T(Z)$:

$$\begin{aligned} \overline{T}_i &= \{(\text{id}, \dots, \text{id}, \overset{i\text{-th component}}{t}, \text{id}, \dots, \text{id})\text{id} \in T(Y) \text{ wr } T(Z) \mid t \in T(Y)\}, \\ \overline{S}_i &= \{(\text{id}, \dots, \text{id}, \overset{i\text{-th component}}{s}, \text{id}, \dots, \text{id})\text{id} \in T(Y) \text{ wr } T(Z) \mid s \in S(Y)\}, \\ \overline{T(Z)} &= \{(\text{id}, \dots, \text{id})t \in T(Y) \text{ wr } T(Z) \mid t \in T(Z)\}, \\ \overline{S(Z)} &= \{(\text{id}, \dots, \text{id})s \in T(Y) \text{ wr } T(Z) \mid s \in S(Z)\}. \end{aligned}$$

Let us first prove (9). As $\alpha \in T(Y) \text{ wr } S(Z)$, we find that $\langle S(Y) \text{ wr } S(Z) \cup \{\alpha\} \rangle \leq T(Y) \text{ wr } S(Z)$. Since $T(Y) = \langle S(Y) \cup \{\bar{\alpha}\} \rangle$, we obtain that $\overline{T}_1 = \langle \overline{S}_1 \cup \{\alpha\} \rangle$, and hence $\overline{T}_1 \leq \langle S(Y) \text{ wr } S(Z) \cup \{\alpha\} \rangle$. For all $i \in \{1, \dots, m\}$, there is some $r \in S(Z)$ such that $1r = i$, and, as discussed before Lemma 2.1, we obtain that $r^{-1}\overline{T}_1r = \overline{T}_i$. Therefore $\overline{T}_1, \dots, \overline{T}_m \leq \langle S(Y) \text{ wr } S(Z) \cup \{\alpha\} \rangle$. As $\overline{S(Z)} \leq \langle S(Y) \text{ wr } S(Z) \cup \{\alpha\} \rangle$ and $T(Y) \text{ wr } S(Z) = (\overline{T}_1 \times \dots \times \overline{T}_m) \rtimes \overline{S(Z)}$, we obtain that $T(Y) \text{ wr } S(Z) \leq \langle S(Y) \text{ wr } S(Z) \cup \{\alpha\} \rangle$, and so the required equality holds.

Now we show that (10). As $\beta \in S(Y) \text{ wr } T(Z)$, we have that $\langle S(Y) \text{ wr } S(Z) \cup \{\beta\} \rangle \leq S(Y) \text{ wr } T(Z)$. As $T(Y) = \langle S(Y) \cup \{\bar{\alpha}\} \rangle$, we obtain that $\overline{T(Z)} = \langle \overline{S(Z)} \cup \{\beta\} \rangle$. As $S(Y) \text{ wr } T(Z) = (\overline{S}_1 \times \dots \times \overline{S}_m) \rtimes \overline{T(Z)}$, we obtain that $S(Y) \text{ wr } T(Z) \leq \langle S(Y) \text{ wr } S(Z) \cup \{\beta\} \rangle$, and so the claim is proved.

As $T(Y) \text{ wr } T(Z) = (\overline{T}_1 \times \dots \times \overline{T}_m) \rtimes \overline{T(Z)}$, the arguments in the previous two paragraphs show that $T(Y) \text{ wr } T(Z) \leq \langle S(Y) \text{ wr } S(Z) \cup \{\alpha, \beta\} \rangle$. As $\alpha, \beta \in T(Y) \text{ wr } T(Z)$, we obtain (11).

As $S(Y) \text{ wr } S(Z)$ is a proper submonoid of each of the monoids $T(Y) \text{ wr } T(Z)$, $T(Y) \text{ wr } S(Z)$ and $S(Y) \text{ wr } T(Z)$, equation (8) must be valid. In order to show (7), it suffices to prove that $\text{rank}(T(Y) \text{ wr } T(Z) : S(Y) \text{ wr } S(Z)) > 1$. Suppose that $\gamma \in T(Y) \text{ wr } T(Z)$ such that $\langle S(Y) \text{ wr } S(Z) \cup \{\gamma\} \rangle = T(Y) \text{ wr } T(Z)$. Then there are $g, g_1, \dots, g_k \in S(Y) \text{ wr } S(Z)$, such that $g\gamma g_1\gamma \dots \gamma g_k = \alpha$. Thus $\gamma g_1\gamma \dots \gamma g_k = g^{-1}\alpha$ and hence $\ker \gamma \subseteq \ker g^{-1}\alpha = \{((1, 1)g, (2, 1)g)\} \cup \Delta$, where $\Delta = \{(x, x) \mid x \in X\}$. Since $\gamma \notin S(Y) \text{ wr } S(Z)$, we obtain that $\ker \gamma = \{((1, 1)g, (2, 1)g)\} \cup \Delta$. Similarly, there exist $h, h_1, \dots, h_k \in S(Y) \text{ wr } S(Z)$ such that $h\gamma h_1\gamma \dots \gamma h_k = \beta$. Thus $\gamma h_1\gamma \dots \gamma h_k = h^{-1}\beta$ and hence

$$\ker \gamma \subseteq \ker h^{-1}\beta = \{((1, 1)h, (1, 2)h), \dots, ((n, 1)h, (n, 2)h)\} \cup \Delta.$$

Hence, for some $i \in \{1, \dots, n\}$, we have $((1, 1)g, (2, 1)g) = ((i, 1)h, (i, 2)h)$, that is, $((1, 1), (2, 1))gh^{-1} = ((i, 1), (i, 2))$. This, however, is a contradiction, because $(1, 1) \sim_{\mathcal{P}} (2, 1)$, but $(i, 1) \not\sim_{\mathcal{P}} (i, 2)$ and the transformation gh^{-1} preserves the equivalence relation $\sim_{\mathcal{P}}$.

Therefore we verified that $\langle S(Y) \text{ wr } S(Z) \cup \{\gamma\} \rangle$ is a proper submonoid of the wreath product $T(Y) \text{ wr } T(Z)$, which shows that equation (7) must hold. \square

4. THE RANK OF $S(X, \mathcal{P})$

In this section we prove the following theorem.

Theorem 4.1. *If X is a finite set such that $|X| \geq 3$ and \mathcal{P} is a uniform partition of X then $S(X, \mathcal{P})$ is generated by two elements.*

In our terminology, the previous theorem gives that $\text{rank } S(X, \mathcal{P}) = 2$. We note that the rank of a transitive permutation group, such as $S(X, \mathcal{P})$, is defined in permutation group theory as the number of orbits of a point-stabilizer. Thus, in order to avoid possible confusion, we decided to state the theorem above without using the notation $\text{rank } S(X, \mathcal{P})$.

Theorem 1.1 will follow from Lemmas 3.1 and 3.2 and Theorem 4.1.

Using the fact that $S(X, \mathcal{P})$ is isomorphic to a wreath product $S(Y) \text{ wr } S(Z)$, it is not difficult to see that $S(X, \mathcal{P})$ is generated by four elements. Indeed, consider the subgroups $\overline{S(Y)}_i$ and $\overline{S(Z)}$ defined in (3) and (4). Since $S(Y) \text{ wr } S(Z) = (\overline{S(Y)}_1 \times \dots \times \overline{S(Y)}_m) \rtimes S(Z)$ and $S(Z)$ is transitive by conjugation on the subgroups $\overline{S(Y)}_i$, we obtain that $S(Y) \text{ wr } S(Z) = \langle \overline{S(Y)}_1, \overline{S(Z)} \rangle$. Since $\overline{S(Y)}_1$ and $\overline{S(Z)}$ are full symmetric groups, they are generated by two elements, and so we obtain that $S(Y) \text{ wr } S(Z)$ is generated by at most four elements. Essentially this is proved in [17, Theorem 2.6].

Before proving Theorem 4.1, we state two simple lemmas.

Let G be a permutation group acting on the set $\{1, \dots, n\}$ and let \mathbb{F} be a field. Let V denote the n -dimensional vector space over \mathbb{F} with basis $\{e_1, \dots, e_n\}$. The group G can be thought of as a permutation group on the set $\{e_1, \dots, e_n\}$, and this defines an $\mathbb{F}G$ -module structure on V as follows:

$$e_i g = e_{ig} \quad \text{for } i \in \{1, \dots, n\} \text{ and } g \in G.$$

The module V is called the *permutation module* for G over \mathbb{F} .

The following lemma is well-known; see, for instance, [18, Lemma 5.3.4].

Lemma 4.1. *If $X = \{1, \dots, n\}$, then the permutation module for $S(X)$ over a field \mathbb{F} of characteristic p has precisely two proper non-trivial submodules:*

$$\begin{aligned} U_1 &= \{(a, a, \dots, a) \mid a \in \mathbb{F}\} \quad \text{and} \\ U_2 &= \{(a_1, \dots, a_n) \mid a_1 + \dots + a_n = 0\}. \end{aligned}$$

Further, if $p \mid n$ then $U_1 \leq U_2$; otherwise $V = U_1 \oplus U_2$.

Suppose that $G = G_1 \times \dots \times G_k$ where the G_i are finite groups. For $I \subseteq \{G_1, \dots, G_k\}$ the function $\varrho_I : G \rightarrow \prod_{G_i \in I} G_i$ is the natural projection map. We also write ϱ_i for $\varrho_{\{G_i\}}$. A subgroup X of G is said to be a *strip* if for each $i = 1, \dots, k$ either $X\varrho_i = 1$ or $X\varrho_i \cong X$. The set of G_i such that $X\varrho_i \neq 1$ is called the *support* of X and is denoted $\text{Supp } X$. Two strips X_1 and X_2 are *disjoint* if $\text{Supp } X_1 \cap \text{Supp } X_2 = \emptyset$. A strip X is said to be *full* if $X\varrho_i = G_i$ for all $G_i \in \text{Supp } X$,

and X is called *non-trivial* if $|\text{Supp } X| \geq 2$. A subgroup K of G is said to be *subdirect* if $K \varrho_i = G_i$ for all i .

We recall a well-known lemma on finite simple groups which can be found in [23, page 328]. The proof of the lemma is elementary and does not use the finite simple group classification.

Lemma 4.2. *Let M be a direct product of finitely many non-abelian, finite simple groups and let H be a subdirect subgroup of M . Then H is the direct product of pairwise disjoint full strips of M .*

The wreath product of two transformation semigroups S and R was defined in Section 2, and the definition can also be used to construct the wreath product of two permutation groups G and H . Recall that the wreath product $G \wr H$ is isomorphic to $G^m \rtimes H$ and a typical element of $G \wr H$ is denoted by $(\pi_1, \dots, \pi_m)\sigma$ where $\pi_i \in G$ and $\sigma \in H$. Setting $\pi = (\pi_1, \dots, \pi_m)$, the same element can also be written as $\pi\sigma$. The following lemma facilitates the calculations in $G \wr H$.

Lemma 4.3. *Let $\pi\sigma, \pi_1\sigma_1, \pi_2\sigma_2 \in G \wr H$ where G and H are as above. Then*

- (i) $\pi_1\sigma_1\pi_2\sigma_2 = \pi_1(\pi_2)^{\sigma_1^{-1}}\sigma_1\sigma_2$;
- (ii) $(\pi\sigma)^{-1} = (\pi^{-1})^{\sigma}\sigma^{-1}$;
- (iii) $(\pi\sigma)^n = \pi\pi^{\sigma^{-1}}\pi^{\sigma^{-2}} \dots \pi^{\sigma^{-n+1}}\sigma^n$.

In particular the projection map $\varrho : G \wr H \rightarrow H$ defined by $\pi\sigma \mapsto \sigma$ is a homomorphism.

Proof. The assertion that ϱ is a homomorphism follows from (i). The rest can be verified using (5) and (6). \square

Now we are ready to prove Theorem 4.1. Permutations of a finite set will be written as products of disjoint cycles.

The proof of Theorem 4.1. By Lemma 2.1(iv), it suffices to show, for $Y = \{1, \dots, n\}$ and $Z = \{1, \dots, m\}$, that the group $W = S(Y) \wr S(Z)$ is generated by two elements whenever $nm \geq 3$. Since, for a finite set Y , the group $S(Y)$ is generated by two elements, we may assume that $n \geq 2$ and $m \geq 2$. Let $A(Y)$ denote the group of even permutations of Y . Then $A(Y)$ is a normal subgroup of index two of $S(Y)$. As $W = S(Y)^m \rtimes S(Z)$, we may consider the subgroups $A(Y)^m$ and $S(Y)^m$ of W and let A and S denote these subgroups respectively.

Let us define

$$\begin{aligned} x &= \begin{cases} (\text{id}, (1, 2), \text{id}, \dots, \text{id})(1, 2, \dots, m) & \text{if either } n \text{ or } m \text{ is odd} \\ (\text{id}, (1, 2), \text{id}, \dots, \text{id})(2, 3, \dots, m) & \text{otherwise} \end{cases} \\ y &= ((1, 2, \dots, n), \text{id}, \dots, \text{id})(1, 2). \end{aligned}$$

Set $M = \langle x, y \rangle$ and we claim that $M = W$.

Let $\varrho : W \rightarrow S(Z)$ denote the natural projection map in Lemma 4.3. If either n or m is odd then $\langle (1, 2), (1, 2, \dots, m) \rangle \leq M\varrho$; otherwise $\langle (1, 2), (2, 3, \dots, m) \rangle \leq M\varrho$. As

$$\langle (1, 2), (1, 2, \dots, m) \rangle = \langle (1, 2), (2, 3, \dots, m) \rangle = S(Z),$$

we obtain that $M\varrho = S(Z)$. As $S = \ker \varrho$, in order to prove that $M = W$, it suffices to show that $S \leq M$.

Next we claim that $A \leq M$. If $n = 2$, then $A = 1$, and so we may assume that $n \geq 3$. First we suppose that $n = 3$. In this case $A \cong (C_3)^m$ and so A can be viewed as a W -module over \mathbb{F}_3 , and, in particular, it can be viewed as a

$S(Z)$ -module over the same field. In fact, A is the natural permutation module for $S(Z)$. Since $M_\varrho = S(Z)$, the intersection $A \cap M$ is an $S(Z)$ -submodule of A . Now $y^2 = ((1, 2, 3), (1, 2, 3), \text{id}, \dots, \text{id})$ and so $y^2 \in A \cap M$, but, if $m \geq 3$, then y^2 is not an element of either of the two proper submodules listed in Lemma 4.1. Therefore $A \cap M = A$, and so $A \leq M$ follows when $n = 3$ and $m \geq 3$. The case $(n, m) = (3, 2)$ can be checked using a computer algebra package such as GAP [8] or MAGMA [6].

Let us assume that $n = 4$, that is, $Y = \{1, \dots, 4\}$, and, as above, we may also assume without loss of generality that $m \geq 3$. Note that $A(Y)$ admits the decomposition $A(Y) = U \rtimes V$ where $U = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$ and $V = \langle (1, 2, 3) \rangle$. Further, U can be considered as an irreducible V -module over \mathbb{F}_2 .

Let m be odd and set $z_1 = (x^m y^2)^2$. As

$$x^m y^2 = ((1, 3, 4), (1, 3, 4), (1, 2), \dots, (1, 2)),$$

we obtain that $z_1 = (x^m y^2)^2 = ((1, 4, 3), (1, 4, 3), \text{id}, \dots, \text{id})$. As $M_\varrho = S(Z)$, the subgroup M has an element of the form $\pi(2, 3)$ where $\pi \in S$. Then set $z = z_1^{\pi(2, 3)}$ and compute that $z = z_1^{\pi(2, 3)} = (\sigma_1, \text{id}, \sigma_3, \text{id}, \dots, \text{id})$ where σ_1 and σ_3 are three-cycles in $S(Y)$. Now set

$$w_1 = (x^m)^{y^4} x^m = ((1, 2)(3, 4), (1, 2)(3, 4), \text{id}, \dots, \text{id})$$

and consider the element $w = w_1^z w_1$. Now, as the first component of $z \in S$ is a non-trivial three cycle, the element w is of the form $w = (\sigma, \text{id}, \dots, \text{id})$ where $\sigma \in \{(1, 2)(3, 4), (1, 3)(4, 2), (1, 4)(2, 3)\}$.

Assume now that m is even and set $z = (x^{m-1} y^2)^4$. Easy computation yields that

$$z = (x^{m-1} y^2)^4 = (\text{id}, (1, 3, 4), \text{id}, \dots, \text{id})$$

and that

$$w = (x^{m-1})^{y^4} x^{m-1} = (\text{id}, (1, 2)(3, 4), \text{id}, \dots, \text{id}).$$

As V is irreducible on U , the computation above shows that the $\langle z \rangle$ -submodule generated by w coincides with $U \times 1 \times \dots \times 1$ if m is odd, and $1 \times U \times 1 \times \dots \times 1$ if m is even. Thus $U \times 1 \times \dots \times 1 \leq M$ in the former case, and $1 \times U \times 1 \times \dots \times 1 \leq M$ in the latter. As $M_\varrho = S(Z)$, we also obtain that $U^m \leq M$.

Now the quotient $A/U^m \cong V^m \cong (C_3)^m$ can be considered as a permutation module for $S(Z)$ over \mathbb{F}_3 , and as $(M \cap A)/U^m \trianglelefteq M/U^m$ and $M_\varrho = S(Z)$, we obtain that $(M \cap A)/U^m$ is a $S(Z)$ -submodule. However, the image of the element z above is not in either of the proper submodules listed in Lemma 4.1, and so we obtain that $(M \cap A)/U^m = A/U^m$ which shows that $A \leq M$.

Hence we have shown that $A \leq M$ if $n \leq 4$. Assume now that $n \geq 5$. In this case $A(Y)$ is a non-abelian finite simple group and so A is a non-abelian characteristically simple group.

Set $Q = M \cap S$. Clearly, $Q \leq S$ and $Q \trianglelefteq M$. Let us show that Q is a subdirect subgroup of $S = S(Y)^m$. If n or m is odd then set $z = x^m = ((1, 2), (1, 2), \dots, (1, 2))$; otherwise set $z = x^{m-1} = (\text{id}, (1, 2), (1, 2), \dots, (1, 2))$. Further,

$$y^2 = ((1, 2, \dots, n), (1, 2, \dots, n), \text{id}, \dots, \text{id}).$$

Therefore $z, y^2 \in Q$. Let ϱ_2 denote the second coordinate projection $\varrho_2 : S \rightarrow S(Y)$. Since $z\varrho_2 = (1, 2) \in Q\varrho_2$ and $y^2\varrho_2 = (1, 2, \dots, n) \in Q\varrho_2$, we obtain that $Q\varrho_2 =$

$S(Y)$. Simple computation shows that if $\tau \in Q$ and $\pi\sigma \in M$ then $\tau^{\pi\sigma}\varrho_{2\sigma} = \tau^\pi\varrho_2$. Therefore

$$S(Y) = Q\varrho_2 = Q^{\pi\sigma}\varrho_{2\sigma} = Q\varrho_{2\sigma}.$$

As $Q\varrho_2 = S(Y)$, it follows that $Q\varrho_i = S(Y)$ for all i , and so Q is a subdirect subgroup of $S = S(Y)^m$. Consider the commutator subgroup Q' . Since $S' = A$, we obtain that $Q' \leq A$. Further, as $Q\varrho_i = S(Y)$, we also obtain, for all i , that $Q'\varrho_i = A(Y)$. Therefore Q' is a subdirect subgroup of $A = A(Y)^m$. Set $R = M \cap A$. Since $Q' \leq R$ and Q' is a subdirect subgroup, we obtain that so is R ; that is, by Lemma 4.2, R is a direct product of disjoint strips.

Let S be a strip in R and let $S \subseteq \{1, \dots, m\}$ be the support of S . We claim that S is a block for the action of $S(Z)$. Indeed, if $\sigma \in S(Z)$ then there is some $\pi \in S$ such that $\pi\sigma \in M$. Then $S^{\pi\sigma}$ is strip in R and so either $S = S^{\pi\sigma}$ or $S \cap S^{\pi\sigma} = 1$. The support of $S^{\pi\sigma}$ is S^σ . Thus either $S^\sigma = S$ or $S^\sigma \cap S = \emptyset$, which shows that S is a block, as required. Since $S(Z)$ is primitive on $\{1, \dots, m\}$ either $|S| = 1$ or $S = \{1, \dots, m\}$. If the latter holds, then Q is a strip. This, however, is impossible. Indeed, if $m = 2$ and n is odd, then set $z = xy = (\text{id}, (1, 2)(1, 2, \dots, n))$; if $m = 2$ and n is even then set $z = xy^2 = ((1, 2, \dots, n), (1, 2)(1, 2, \dots, n))$; if $m \geq 3$, then set $z = y^2 = ((1, 2, \dots, n), (1, 2, \dots, n), \text{id}, \dots, \text{id})$. Then in all cases $z^2 \in R$, but z^2 is not in a full strip (if $m = 2$ and n is even then note that the first component of z^2 is of order $n/2$ and the second is of order $n - 1$). Thus $|S| = 1$, and so $A \leq M$.

This completes the proof of the claim that that $A \leq M$ for all n and m .

Note that A is a normal subgroup of W and let $x \mapsto \hat{x}$ denote the natural homomorphism $W \rightarrow W/A$. If $H \leq W$, then \hat{H} denotes the image HA/A of H . Then $\widehat{W} \cong C_2 \text{ wr } S(Z)$ and $\widehat{S} \cong (C_2)^m$. We claim that $\widehat{S} \leq \widehat{M}$. Note that $\widehat{M} \cap \widehat{S} \trianglelefteq \widehat{M}$. If $\pi\sigma \in \widehat{M}$ and $u \in \widehat{M} \cap \widehat{S}$ then $u^{\pi\sigma} = u^\sigma$. As $u^{\pi\sigma} \in \widehat{M} \cap \widehat{S}$ we obtain that $u^\sigma \in \widehat{M} \cap \widehat{S}$ which shows that $\widehat{M} \cap \widehat{S}$ is a $S(Z)$ -submodule of $\widehat{S} \cong (C_2)^m$. It is clear that \widehat{S} is the natural permutation module for $S(Z)$ over \mathbb{F}_2 . Lemma 4.1 lists the non-trivial proper submodules U_1 and U_2 of \widehat{S} .

If both n and m are even then $\hat{x}^{m-1} = (0, 1, \dots, 1)$ and this element is not in either U_1 or U_2 . Hence $\widehat{S} \leq \widehat{M}$. If this is not the case, then $\hat{x}^m = (1, 1, \dots, 1)$ which shows that $U_1 \leq \widehat{M}$. If n is even and m is odd then $\hat{y}^2 = (1, 1, 0, \dots, 0)$ and so $U_2 \leq \widehat{M}$. Therefore in this case $U_1 \oplus U_2 \leq \widehat{M}$ follows.

Suppose that n is odd. If $m = 2$ then $xy = (\text{id}, (1, 2)(1, 2, \dots, n))$. Thus \widehat{xy} is not in either of the proper submodules listed in Lemma 4.1. Hence $\widehat{S} \leq \widehat{M}$ follows in this case. If $m \geq 3$, then

$$xy = (\text{id}, (1, 2), \text{id}, \dots, \text{id}, (1, 2, \dots, n))(2, 3, \dots, m).$$

Let $\pi = (\text{id}, (1, 2), \text{id}, \dots, \text{id}, (1, 2, \dots, n))$ and $\sigma = (2, 3, \dots, m)$ so that $xy = \pi\sigma$. As $\sigma^{m-1} = 1$, it follows from Lemma 4.3 that

$$(xy)^{m-1} = \pi\pi^{\sigma^{-1}} \dots \pi^{\sigma^{-m+2}}.$$

Thus $(xy)^{m-1}$ is of the form $(\text{id}, \pi_2, \dots, \pi_m)$ where, for $i = 2, \dots, m$, the permutation π_i is either $(1, 2)(1, 2, \dots, n)$ or $(1, 2, \dots, n)(1, 2)$, that is, π_i is a cycle with length $n - 1$. As n is odd, $\pi_i \notin A(Y)$, and so $(\widehat{xy})^{m-1} = (0, 1, 1, \dots, 1)$. Now if m is even then $(\widehat{xy})^{m-1}$ is not an element of U_1 or U_2 , and so $\widehat{S} \leq \widehat{M}$ follows also in this case. If m is odd then this shows that $U_2 \leq \widehat{M}$, and as we proved above that $U_1 \leq \widehat{M}$, it follows that $\widehat{S} \leq \widehat{M}$.

Thus we have shown that $S \leq M$ as required. As explained above, $M = W$ now follows. \square

The main result of the paper can now be proved.

The proof of Theorem 1.1. As usual, we assume without loss of generality that $X = Y \times Z$ where $Y = \{1, \dots, n\}$, $Z = \{1, \dots, m\}$ and \mathcal{P} is the partition

$$\{\{(1, 1), \dots, (n, 1)\}, \dots, \{(1, m), \dots, (n, m)\}\}.$$

By Lemmas 2.1, 3.1, 3.2 and Theorem 4.1,

$$\begin{aligned} \text{rank } T(X, \mathcal{P}) &= \text{rank } (T(X, \mathcal{P}) : S(X, \mathcal{P})) + \text{rank } S(X, \mathcal{P}) \\ &= \text{rank } (T(Y) \text{ wr } T(Z) : S(Y) \text{ wr } S(Z)) + \text{rank } S(Y) \text{ wr } S(Z) = 4. \end{aligned}$$

The assertions concerning $\text{rank } \Gamma(X, \mathcal{P})$ and $\text{rank } \Sigma(X, \mathcal{P})$ can be proved very similarly. \square

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(Schneider) INFORMATICS RESEARCH LABORATORY, COMPUTER AND AUTOMATION RESEARCH
INSTITUTE, 1518 BUDAPEST PF. 63, HUNGARY
CENTRO DE ÁLGEBRA DA UNIVERSIDADE DE LISBOA, AV. PROF. GAMA PINTO 2, 1649-003 LISBOA,
PORTUGAL
EMAIL: CSABA.SCHNEIDER@GMAIL.COM
WWW: WWW.SZTAKI.HU/~SCHNEIDER

(Araújo) UNIVERSIDADE ABERTA, R. ESCOLA POLITÉCNICA 147, 1269-001 LISBOA, PORTUGAL
CENTRO DE ÁLGEBRA, UNIVERSIDADE DE LISBOA, 1649-003 LISBOA, PORTUGAL
EMAIL: JARAUJO@LMC.FC.UL.PT